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Abstract. A set of coherent states of the semi-direct product $N(2) \otimes Sp(4, R)$ of the Heisenberg–Wyl and symplectic groups is constructed; they are shown to be non-spreading two-dimensional wavepackets moving on closed classical trajectories. These states generalise in a natural manner the Glauber coherent states to the description, in the classical limit, of a particle in a rotating frame as well as of a charged particle in a uniform magnetic field. The properties of these states are studied.

1. Introduction

Since its introduction by Schrödinger (1926) and its development and application in quantum optics by Glauber (1963), the oscillator or equivalently the Heisenberg–Wyl coherent states have found widespread application in the description of collective and cooperative phenomena in various fields of physics. Motivated by such developments, Radcliff (1971) and Block (1946) constructed a set of so-called spin coherent states and hoped that they would find applications in problems involving spins and their correlations. These states have subsequently been studied and used by a number of authors (see Perelomov (1977) and references therein). Moreover, they have been generalised to those of the group SU(1, 1) by Barut and Girardello (1971), Hongoh (1976) and Berghe and De Meyer (1978). Their generalisation to the rotation groups and the group of an asymmetric top has been given by Mikhailov (1973) and Janssen (1977) (see also Gulshani (1979a, b) and references therein).

However, the construction of the coherent states of the rotation groups and those of an asymmetric top given by the author (Gulshani 1979a, b) is based on Schwinger's boson realisation of the angular momentum algebra. This is an abstract realisation and is not physical, in the sense that the basic physical variables, the positions and momenta of the particles of the system, do enter into consideration. As a result, the coherent states thereby constructed are not directly related to the microscopic structure of the system they purport to describe. It is, therefore, highly desirable to obtain a set of angular momentum coherent states which are manifestly microscopic, meaning that they are explicitly dependent on the basic position observables. The importance of such a construction lies in the desire to understand from first principles and in a quantum setting the rotational motion of a self-bound system of particles, such as a nucleus (Bohr *et al* 1976).

In this paper such a set of angular momentum coherent states in two dimensions and for a system of a single particle is constructed. This is a subset of the coherent states associated with the semi-direct product group N(2) ⊗ Sp(4, R) of the Heisenberg–Wyl and symplectic groups N(2) and Sp(4, R) respectively. They are shown to generalise the Glauber coherent states (Glauber 1963) in a natural way. In § 2 the harmonic oscillator dynamical group N(2) ⊗ Sp(4, R) is reviewed. The associated angular momentum coherent states are constructed in § 3 and their properties are studied in detail in § 4. They are shown to be two-dimensional non-dispersive wavepackets with their centres moving on rather arbitrary closed classical trajectories in two dimensions. A special case of these angular momentum coherent states is shown to be those associated with the motion of a charged particle in a uniform magnetic field. This paper is, then, concluded in § 5 with a few suggested uses for these angular momentum coherent states, for example in the nuclear cranking model. In § 5 we also mention the possible generalisations of these angular momentum coherent states to those associated with the full group N(2) ⊗ Sp(4, R). These states describe both rotational and vibrational motions as well as their intercoupling. A further, perhaps more interesting, generalisation is to three dimensions, where the appropriate group is naturally the semi-direct product group N(3) ⊗ Sp(6, R), the dynamical group of the three-dimensional oscillator.

2. N(2) ⊗ Sp(4, *R*) as oscillator dynamical group

It is well known (Wybourne 1974, Major 1977a, b) that the semi-direct product group N(2) ⊗ Sp(4, R) of the Heisenberg–Wyl N(2) and the symplectic Sp(4, R) groups is a dynamical group of the harmonic oscillator in two dimensions. The Lie algebra of N(2) is spanned by the set $\{x_k, p_k, I;\ k=1,2\}$ where x_k, p_k are the position and momentum coordinates respectively and I is the identity. A more convenient basis is its complex extension $\{a_k, a_k^\dagger, I\}$ where the usual oscillator bosons a_k, $a_k^\dagger$ satisfy the commutation relations $[a_k, a_l^\dagger]=\delta_{kl}$, $[a_k, a_l]=[a_k^\dagger, a_l^\dagger]=0$. The Lie algebra of Sp(4, R) is realised by the set of all bilinear products formed from a_k and $a_k^\dagger$ (Goshen and Lipkin 1959, 1968, Lipkin 1966, Moshinsky and Quesne 1971, Wybourne 1974). Unitary realisations of the groups N(2) and Sp(4, R) are then obtained by exponentiating skew adjoint operators of the corresponding Lie algebras. Now the harmonic oscillator states are known to be irreducible under the action of the Heisenberg–Wyl group N(2). These states also span two unitary irreducible representations of the symplectic group, one involving only even and the other only odd parity states (Moshinsky and Quesne 1971, Quesne and Moshinsky 1971). The group N(2), of course, mixes these two irreducible representations. It then follows (Perelomov 1972, 1977) that the coherent states of the group N(2) ⊗ Sp(4, R) are given by the action of the group elements on a fixed two-dimensional harmonic oscillator state.

For the representation space we choose the states of the harmonic oscillator with the Hamiltonian

$$H_0 \equiv \hbar \sum_{k=1}^{2} \alpha_k (a_k^\dagger a_k + \tfrac{1}{2}) \tag{2.1}$$

where, to encompass the anisotropic as well as the isotropic oscillator, the frequencies α_k are taken to be independent of each other. The boson operators a_k and $a_k^\dagger$ are then defined by

$$a_k^\dagger \equiv \left(\frac{m\alpha_k}{2\hbar}\right)^{1/2}\left(x_k - \frac{\mathrm{i}}{m\alpha_k}p_k\right) \qquad \text{and} \qquad a_k \equiv \left(\frac{m\alpha_k}{2\hbar}\right)^{1/2}\left(x_k + \frac{\mathrm{i}}{m\alpha_k}p_k\right)$$

where m is the particle mass. As is well known the energy eigenvalues and eigenstates of H_0 are respectively

$$E_{n_1n_2} = E_{n_1} + E_{n_2} = \hbar \sum_{k=1}^{2} \alpha_k(n_k + \tfrac{1}{2}) \tag{2.2}$$

and

$$|n_1n_2\rangle = \frac{(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}}{(n_1!n_2!)^{1/2}}|00\rangle \tag{2.3}$$

where $|00\rangle$ is the vacuum of a_k, i.e. $a_k|00\rangle = 0$.

The class of coherent states obtained from the action of the full group $\mathrm{N}(2) \otimes \mathrm{Sp}(4, \boldsymbol{R})$ on a fixed state in equation (2.3) is rather large, describing rotational as well as monopole and quadrupole vibrational modes of motion. In this paper, we consider only the subclass of these coherent states associated with the rotational degree of freedom.

3. N(2) ⊗ Sp(4, *R*) angular momentum coherent states

We begin with the generalised Glauber coherent states for the two-dimensional anisotropic oscillator system defined in equations (2.1)–(2.3). These states are defined by (Glauber 1963, Perelomov 1977)

$$|A_1, A_2; n_1n_2\rangle \equiv D(A_1)D(A_2)|n_1n_2\rangle \tag{3.1}$$

where the displacement operator D is an element of N(2),

$$D(A_k) \equiv \exp(A_k a_k^\dagger - A_k^* a_k), \qquad k = 1, 2, \tag{3.2}$$

and A_k are two arbitrary complex amplitudes. Consider now the particular unitary element

$$U(\gamma_j) \equiv \exp\left(-\frac{\mathrm{i}\gamma_1}{\hbar}x_1x_2\right) \exp\left(-\frac{\mathrm{i}\gamma_2}{\hbar}p_1p_2\right)$$
$$* \exp -\frac{\mathrm{i}}{2\hbar}[\ln \gamma_3(x_1p_1 + p_1x_1) + \ln \gamma_4(x_2p_2 + p_2x_2)] \tag{3.3}$$

of the group $\mathrm{Sp}(4, \boldsymbol{R})$, where γ_j ($k = 1, 2, 3, 4$) are arbitrary real parameters and where for convenience we have chosen to write U in terms of the position and the momentum coordinates x_k and p_k rather than the boson operators a_k, $a_k^\dagger$. It is easy to check that the set $\{x_1x_2, p_1p_2, \frac{1}{2}(x_1p_1 + p_1x_1), \frac{1}{2}(x_2p_2 + p_2x_2)\}$ is closed under commutation relations and hence spans a four-dimensional subalgebra of $\mathrm{sp}(4, \boldsymbol{R})$†. It is, therefore, a simple matter, by means of the method of parameter differentiation (Wilcox 1967), to find the law of composition for the elements of this subgroup of $\mathrm{Sp}(4, \boldsymbol{R})$.

We now define the $\mathrm{N}(2) \otimes \mathrm{Sp}(4, \boldsymbol{R})$ angular momentum coherent states in two dimensions by

$$|A_1, A_2; \gamma_j, n_1n_2\rangle \equiv U(\gamma_j)|A_1, A_2; n_1n_2\rangle = U(\gamma_j)D(A_1)D(A_2)|n_1n_2\rangle \tag{3.4}$$

where $|A_1, A_2; n_1n_2\rangle$ are the generalised Glauber coherent states in equation (3.1).

† In fact the subset $\{x_1x_2, p_1p_2, x_1p_1 + p_2x_2\}$ is isomorphic to the non-unitary complex extension of $\mathrm{so}(3) \sim \mathrm{so}(2, 1) \sim \mathrm{su}(1, 1)$.

Again using the methods of parameter differentiation (Wilcox 1967), one can easily combine the products of the unitary operators in (3.4) into a single element of the group $N(2) \otimes Sp(4, R)$. However, it is seen that the coherent states in (3.4) are unitarily related to the Glauber coherent states and as such have the same orthogonality and completeness properties as the latter, namely

$$\langle A_1, A_2; \gamma_j; n_1 n_2 | B_1, B_2; \gamma_j; n_1 n_2 \rangle$$
$$= \prod_{i=1}^{2} \exp -\tfrac{1}{2}(|A_i|^2 + |B_i|^2 - 2A_i^* B_i) * \sum_{l=0}^{n_i} \frac{|B_i - A_i|^{2l}}{l!} \binom{n_i}{l}$$

and

$$\frac{1}{\pi^2} \int d^2 A_1 \, d^2 A_2 | A_1, A_2; \gamma_j; n_1 n_2 \rangle \langle A_1, A_2; \gamma_j; n_1 n_2 | \equiv 1$$

where $d^2 A_k \equiv d(\mathrm{Re}\, A_k)\, d(\mathrm{Im}\, A_k)$, $(k = 1, 2)$. This resolution of unity follows immediately from the fact (Perelomov 1977) that

$$\frac{1}{\pi^2} \int d^2 A_1 \, d^2 \mathrm{A}_2 | A_1, A_2; n_1 n_2 \rangle \langle A_1, A_2; n_1, n_2 | \equiv 1.$$

The time evolution of the coherent states (3.4) will be studied with respect to the transformed Hamiltonian

$$H \equiv U H_0 U^\dagger \equiv \hbar \sum_{k=1}^{2} \alpha_k (\tilde{a}_k^\dagger \tilde{a}_k + \tfrac{1}{2}) \tag{3.5}$$

where H_0 is defined in equation (2.1) and

$$\tilde{a}_k^\dagger \equiv U a_k^\dagger U^\dagger, \qquad \tilde{a}_k \equiv U a_k U^\dagger. \tag{3.6}$$

The Hamiltonian H satisfies the Schrödinger equation

$$H | \tilde{n}_1 \tilde{n}_2 \rangle \equiv E_{n_1 n_2} | \tilde{n}_1 \tilde{n}_2 \rangle$$

where

$$| \tilde{n}_1 \tilde{n}_2 \rangle \equiv U | n_1 n_2 \rangle \tag{3.7}$$

and $E_{n_1 n_2}$ and $|n_1 n_2\rangle$ are defined in equations (2.2) and (2.3) respectively. The development in time of the $N(2) \otimes Sp(4, R)$ angular momentum coherent states in (3.4) wrt the Hamiltonian H is then given by†

$$|A_1(t), A_2(t); \gamma_j; n_1 n_2 \rangle$$
$$= \exp[-(\mathrm{i}t/\hbar)H] | A_1, A_2; \gamma_j; n_1 n_2 \rangle$$
$$= U(\gamma_j) \exp[-(\mathrm{i}t/\hbar)H_0] D(A_1) D(A_2) | n_1 n_2 \rangle$$
$$= \exp[-(\mathrm{i}t/\hbar)E_{n_1 n_2}] U(\gamma_j) D(A_1(t)) D(A_2(t)) | n_1 n_2 \rangle$$
$$= \exp[-(\mathrm{i}t/\hbar)E_{n_1 n_2}] \tilde{D}(A_1(t)) \tilde{D}(A_2(t)) | \tilde{n}_1 \tilde{n}_2 \rangle \tag{3.8}$$

where in (3.8) we have used equations (3.5) and (3.2), the well-known identity

$$\exp[-(\mathrm{i}t/\hbar)H_0] a_k^\dagger \exp[(\mathrm{i}t/\hbar)H_0] = a_k^\dagger \exp(-\mathrm{i}\alpha_k t)$$

† With respect to H_0 in equation (2.1), these coherent states have complicated time development and will not have the simple rotational properties of the states in (3.8).

and the definitions (3.6) and

$$A_k(t) \equiv A_k \exp(-\mathrm{i}\alpha_k t) \tag{3.9}$$

$$\tilde{D}(A_k) \equiv UD(A_k)U^\dagger = \exp(A_k\tilde{a}_k^\dagger - A_k^*\tilde{a}_k^\dagger). \tag{3.10}$$

4. Properties of N(2) ⊗ Sp(4, *R*) angular momentum coherent states

In the previous section we discussed the orthogonality and the completeness of the coherent states in equations (3.8) or (3.4). The expansion of an arbitrary function, such as the state of good angular momentum, in terms of these states will not be considered in this paper. But we would like to consider now the expectation values and the variances in the states (3.8) of the various physically interesting observables. For this purpose one needs to know the action of the unitary operator U in (3.3) on x_k and p_k. Clearly U induces a linear canonical transformation in the phase space which may be seen to be of scaling and gauge type. Using the expansion

$$e^x B\, e^{-x} = B + [x, B] + (1/2!)[x, [x, B]] + \dots,$$

one easily finds

$$U^\dagger\begin{pmatrix} x_1 \\ p_2 \end{pmatrix} U = \begin{pmatrix} \gamma_3 & \gamma_2/\gamma_4 \\ -\gamma_1\gamma_3 & (1-\gamma_1\gamma_2)/\gamma_4 \end{pmatrix}\begin{pmatrix} x_1 \\ p_2 \end{pmatrix},$$
$$U^\dagger\begin{pmatrix} x_2 \\ p_1 \end{pmatrix} U = \begin{pmatrix} \gamma_4 & \gamma_2/\gamma_3 \\ -\gamma_1\gamma_4 & (1-\gamma_1\gamma_2)/\gamma_3 \end{pmatrix}\begin{pmatrix} x_2 \\ p_1 \end{pmatrix}. \tag{4.1}$$

The inverse transformation is similarly determined. One obtains a better understanding of the transformation by various specialisation of the parameters γ_j. For instance, it is possible to choose γ_j such that the transformed Hamiltonian H in (3.5) has the form $\tilde{H}_0 + \Omega J_3$ where $\tilde{H}_0$ is some oscillator Hamiltonian, $J_3 \equiv x_1p_2 - x_2p_1$, i.e. the third component of the angular momentum operator, and Ω is a real parameter. It is then seen that $H_0 + \Omega J_3$ is the Hamiltonian H_0 but observed in a frame rotating with the angular frequency $|\Omega|$.

Using equation (4.1) and the well known result (Glauber 1963) $D^\dagger(A_k)a_k^\dagger D(A_k) = a_k^\dagger + A_k^*$, the expectation values, in the coherent states (3.8), of the position coordinates are readily found to be

$$\langle x_1\rangle \equiv \langle A_1(t), A_2(t); \gamma_j; n_1n_2|x_1|A_1(t), A_2(t); \gamma_j; n_1n_2\rangle$$
$$= |A_1|\left(\frac{2\hbar}{m\alpha_1}\right)^{1/2}\gamma_3\cos(\alpha_1 t + \phi_1) - |A_2|(2m\hbar\alpha_2)^{1/2}\frac{\gamma_2}{\gamma_4}\sin(\alpha_2 t + \phi_2), \tag{4.2}$$
$$\langle x_2\rangle = |A_2|\left(\frac{2\hbar}{m\alpha_2}\right)^{1/2}\gamma_4\cos(\alpha_2 t + \phi_2) - |A_1|(2m\hbar\alpha_1)^{1/2}\frac{\gamma_2}{\gamma_3}\sin(\alpha_1 t + \phi_1),$$

where we have defined

$$A_k(t) \equiv |A_k| \exp - \mathrm{i}(\alpha_k t + \phi_k)$$

with $|A_k|$ and ϕ_k being the modulus and the phase of A_k. Equations (4.2) are seen to be identical to those describing the motion of a classical particle which is a superposition of two elliptical orbits in the plane x_1–x_2. The orbits have frequencies α_1 and α_2, the senses of rotations are clockwise and anticlockwise respectively and the axes of the ellipses bear the ratios $1: m\alpha_1\gamma_2/\gamma_3^2$ and $1: \gamma_4^2/m\alpha_2\gamma_2$. We are free to obtain either one of the orbits by setting either $|A_1|=0$ or $|A_2|=0$.

The information on the extent of localisation and dispersiveness of the rotational coherent states in (3.8) is contained in the fluctuations in the values of the various physical observables. The variances of the position and the momentum operators in the coherent states (3.8) are found to be

$$
\begin{aligned}
(\Delta x_1)^2 &\equiv \langle x_1^2\rangle - \langle x_1\rangle^2 = \tfrac{1}{2}\hbar[(1/m\alpha_1)(2n_1+1)\gamma_3^2 + m\alpha_2(2n_2+1)\gamma_2^2/\gamma_4^2],\\
(\Delta x_2)^2 &= \tfrac{1}{2}\hbar[(1/m\alpha_2)(2n_2+1)\gamma_4^2 + m\alpha_1(2n_1+1)\gamma_2^2/\gamma_3^2],\\
(\Delta p_1)^2 &= \frac{\hbar}{2}\left[m\alpha_1(2n_1+1)\left(\frac{1-\gamma_1\gamma_2}{\gamma_3}\right)^2 + \frac{1}{m\alpha_2}(2n_2+1)\gamma_1^2\gamma_4^2\right],\\
(\Delta p_2)^2 &= \frac{\hbar}{2}\left[m\alpha_2(2n_2+1)\left(\frac{1-\gamma_1\gamma_2}{\gamma_4}\right)^2 + \frac{1}{m\alpha_1}(2n_1+1)\gamma_1^2\gamma_3^2\right].
\end{aligned} \tag{4.3}
$$

Equations (4.3) indicate that the coherent states (3.8) are non-dispersive and the uncertainty products $(\Delta x_1)(\Delta p_1)$ and $(\Delta x_2)(\Delta p_2)$ assume the minimum value of $\hbar/2$ for the values $n_1=n_2=0$, $\gamma_1=(1/\beta)\tan\alpha\beta$, $\gamma_2=\beta\cos\alpha\beta\sin\alpha\beta$ and $\gamma_3=\gamma_4=\cos\alpha\beta$ with $\beta=1/m\alpha_1\alpha_2$ and α arbitrary†.

More interesting and important types of fluctuations pertinent to a rotating quantum system are, however, those associated with the angular momentum and the orientation of the system. The reason for this stems from the fact that such a system must be deformed like a molecule (Bohr *et al* 1976). The state of such a system is, therefore, a superposition of states by definite angular momenta. To examine the coherent states (3.8) for these effects consider first the value of the classical angular momentum $\bar{J}_3^{(c)}$ for the motion described in equation (4.2). One finds that

$$
\begin{aligned}
\bar{J}_3^{(c)} &\equiv m\langle x_1\rangle\frac{d}{dt}\langle x_2\rangle - m\langle x_2\rangle\frac{d}{dt}\langle x_1\rangle\\
&= 2\hbar m\gamma_2(\alpha_2|A_2|^2 - \alpha_1|A_1|^2)\\
&\quad + \frac{\hbar}{\gamma_3\gamma_4}\left[\left(\frac{\alpha_2}{\alpha_1}\right)^{1/2}(\gamma_3^2\gamma_4^2 - m^2\alpha_1^2\gamma_2^2) - \left(\frac{\alpha_1}{\alpha_2}\right)^{1/2}(\gamma_3^2\gamma_4^2 - m^2\alpha_2^2\gamma_2^2)\right]\mathrm{Im}(A_1A_2)\\
&\quad + \frac{\hbar}{\gamma_3\gamma_4}\left[\left(\frac{\alpha_2}{\alpha_1}\right)^{1/2}(\gamma_3^2\gamma_4^2 - m^2\alpha_1^2\gamma_2^2) + \left(\frac{\alpha_1}{\alpha_2}\right)^{1/2}(\gamma_3^2\gamma_4^2 - m^2\alpha_2^2\gamma_2^2)\right]\mathrm{Im}(A_1A_2^*).
\end{aligned} \tag{4.4}
$$

Equation (4.4) shows that $\bar{J}_3^{(c)}$ oscillates in time. When $|A_2|=0$, $|A_1|\neq 0$, (4.4) reduces to the constant value of the angular momentum of the clockwise orbit, and likewise when $|A_1|=0$ and $|A_2|\neq 0$. For the case of the isotropic Hamiltonian H_0 in (1.1), $\alpha_1=\alpha_2$, the third term in equation (4.4) vanishes and the fourth term reduces to a constant.

Now the classical angular momentum $\bar{J}_3^{(c)}$ ought to be compared with the expectation value of the mechanical angular momentum operator $\bar{J}_3^{(Q)}$ in the coherent states

† It is interesting to note that, although for these particular values of γ_j $(j=1,2,3,4)$ the minimum uncertainty products are obtained in agreement with the result (Stoler 1970, Carnivell and Segler 1977) that states unitarily related to the minimum packet are minimum packets, this result does not seem to have validity in dimensions higher than one.

(3.8), rather than with that of the canonical angular momentum $J_3^{(Q)} \equiv x_1p_2 - x_2p_1$ (as distinguished by the overhead bars). This is because the coherent states (3.8) are translated in time via the Hamiltonian H in equation (3.5) and not H_0. From equation (3.8) and the definition of $\bar{J}_3^{(c)}$ in (4.4) one easily deduces the definition

$$\bar{J}_3^{(Q)} \equiv mx_1\frac{dx_2}{dt} - mx_2\frac{dx_1}{dt} = \frac{mi}{\hbar}(x_1[H, x_2] - x_2[H, x_1]). \tag{4.5}$$

The expectation value of $\bar{J}_3^{(Q)}$ in the coherent states (3.8) is, then, evaluated to be

$$\langle \bar{J}_3^{(Q)} \rangle = \bar{J}_3^{(c)} + 2m\gamma_2(E_{n_2} - E_{n_1}) \tag{4.6}$$

where E_{n_2} and E_{n_1} are defined in equation (2.2). Obviously, for a given α_1 and α_2 the difference between $\langle \bar{J}_3^{(Q)} \rangle$ and $\bar{J}_3^{(c)}$ is minimum in absolute value when $n_1 = n_2$ and vanishes when $E_{n_1} = E_{n_2}$. It is also observed that the second quantity on the RHS of equation (4.6) is the value of $\langle \bar{J}_3^{(Q)} \rangle$ when the semi-classical nature of the coherent states in (3.8) disappears, i.e. when $|A_1| = |A_2| = 0$. Thus $\langle \bar{J}_3^{(Q)} \rangle$ in equation (4.6) is the sum of the classical and the quantum angular momenta! This result may be understood from the last expression in equation (3.8) and the fact that the operator U, acting on the state $|n_1n_2\rangle$ which has zero mean angular momentum, generates the state $|\tilde{n}_1\tilde{n}_2\rangle$ with non-zero mean angular momentum (cf equations (3.7) and (4.6)).

A measure of sharpness of the angular momentum of the coherent states (3.8) about $\langle \bar{J}_3^{(Q)} \rangle$ is given by the variance $\Delta\bar{J}_3$. Using the expression

$$U^\dagger \bar{J}_3^{(Q)} U = 2m\gamma_2(H_{02} - H_{01}) + \left(\gamma_3\gamma_4 - \frac{m^2\alpha_1^2\gamma_2^2}{\gamma_3\gamma_4}\right)x_1p_2 - \left(\gamma_3\gamma_4 - \frac{m^2\alpha_2^2\gamma_2^2}{\gamma_3\gamma_4}\right)x_2p_1$$

where $H_{02} \equiv \hbar\alpha_2(a_2^\dagger a_2 + \frac{1}{2})$ and $H_{01} \equiv \hbar\alpha_1(a_1^\dagger a_1 + \frac{1}{2})$, one finds that

$$\begin{aligned}
(\Delta\bar{J}_3)^2 &\equiv \langle \bar{J}_3^{(Q)2} \rangle - \langle \bar{J}_3^{(Q)} \rangle^2 \\
&= 8m^2\gamma_2^2(\alpha_2E_{n_2}|A_2|^2 + \alpha_1E_{n_1}|A_1|^2) \\
&\quad - \frac{\hbar^2}{2}\left(\gamma_3\gamma_4 - \frac{m^2\alpha_1^2\gamma_2^2}{\gamma_3\gamma_4}\right)\left(\gamma_3\gamma_4 - \frac{m^2\alpha_1^2\gamma_2^2}{\gamma_3\gamma_4}\right) + \frac{\hbar^2\alpha_2}{4\alpha_1}\left(\gamma_3\gamma_4 - \frac{m^2\alpha_1^2\gamma_1^2}{\gamma_3\gamma_4}\right)^2 \\
&\quad \times [(2n_1+1)(2n_2+1) + 4(2n_2+1)\,\mathrm{Re}^2 A_1 + 4(2n_1+1)\,\mathrm{Im}^2 A_2 \\
&\quad + \frac{\hbar^2\alpha_1}{4\alpha_2}\left(\gamma_3\gamma_4 - \frac{m\alpha_2^2\gamma_2^2}{\gamma_3\gamma_4}\right)^2 \\
&\quad \times [(2n_1+1)(2n_2+1) + 4(2n_2+1)\,\mathrm{Im}^2 A_1 + 4(2n_1+1)\,\mathrm{Re}^2 A_2] \\
&\quad + 8\hbar m_2\gamma_2(E_{n_2} - E_{n_1})\left[\left(\gamma_3\gamma_4 - \frac{m^2\alpha_1^2\gamma_2^2}{\gamma_3\gamma_4}\right)\left(\frac{\alpha_2}{\alpha_1}\right)^{1/2} \mathrm{Re}\,A_1\,\mathrm{Im}\,A_2\right. \\
&\quad \left. - \left(\gamma_3\gamma_4 - \frac{m^2\alpha_1^2\gamma_2^2}{\gamma_3\gamma_4}\right)\left(\frac{\alpha_1}{\alpha_2}\right)^{1/2} \mathrm{Re}\,A_2\,\mathrm{Im}\,A_1\right].
\end{aligned} \tag{4.7}$$

It is seen in (4.7) that $(\Delta\bar{J}_3)^2$ oscillates in time with frequencies α_1 when $|A_2| = 0$, $|A_1| \neq 0$ and with α_2 when $|A_1| = 0$, $|A_2| \neq 0$. For an isotropic oscillator ($\alpha_1 = \alpha_2$) $(\Delta\bar{J}_3)^2$ is, however, stationary. The expression (4.7) is, however, too complicated to make any more useful observations. It does, nevertheless, demonstrate the non-dispersive nature of the angular momentum coherent states (3.8) in the sense that $(\Delta\bar{J}_3)^2$ is oscillatory.

In contrast to the well defined observables dealt with so far there is, on the other hand, no unique and/or well defined operator associated with the orientation of a state in quantum mechanics. Among the numerous classical angle variables conjugate to $J_3^{(Q)}$ that one may define (Lipkin and Goldstein 1958a, b, Verhaar 1962) even the simplest, namely $\tan^{-1}(x_2/x_1)$, is difficult to deal with. There is an additional complication here because we have to deal with $\bar{J}_3^{(Q)}$ rather than $J_3^{(Q)}$. One may, however, dispense with the notion of angle and instead examine a certain multipole distribution of the states in equation (3.8). One may, for instance, look at the quadrupole moment x_1x_2. For this quantity one finds that $\langle x_1x_2\rangle = \langle x_1\rangle\langle x_2\rangle$ and $\langle x_1^2x_2^2\rangle - \langle x_1x_2\rangle^2$ assumes its smallest value for $n_1 = n_2 = 0$. A more satisfying alternative is, however, to look at the time development of the probability density itself in the coordinate space. We now derive expressions for the coherent states (3.8) and the corresponding probability density in the coordinate space for the special case when $n_1 = n_2 = 0$.

When $n_1 = n_2 = 0$, the coherent states in equation (3.8) become simultaneous eigenstates of the commuting annihilation operators $\tilde{a}_1$ and $\tilde{a}_2$ defined in (3.6) because $\tilde{D}^{\dagger}(A_k)\tilde{a}_k\tilde{D}(A_k) = \tilde{a}_k + A_k$ and $\tilde{a}_k|\tilde{0}\tilde{0}\rangle \equiv 0$ (cf equations (3.8), (3.7) and (2.3)). We therefore have

$$\tilde{a}_k|A_1(t), A_2(t); \gamma_j; 00\rangle = A_k(t)|A_1(t), A_2(t); \gamma_j; 00\rangle \tag{4.8}$$

where, using the inverse of the transformation in equation (4.1),

$$\begin{aligned} \tilde{a}_1 &= \left(\frac{m\alpha_1}{2\hbar}\right)^{1/2}\left[\frac{1-\gamma_1\gamma_2}{\gamma_3}x_1 - \frac{\gamma_2}{\gamma_3}p_2 + \frac{\mathrm{i}}{m\alpha_1}(\gamma_1\gamma_3x_2 + \gamma_3p_1)\right], \\ \tilde{a}_2 &= \left(\frac{m\alpha_2}{2\hbar}\right)^{1/2}\left[\frac{1-\gamma_1\gamma_2}{\gamma_4}x_2 - \frac{\gamma_2}{\gamma_4}p_1 + \frac{\mathrm{i}}{m\alpha_2}(\gamma_1\gamma_4x_1 + \gamma_4p_2)\right]. \end{aligned} \tag{4.9}$$

The two first-order linear partial differential equations in (4.8) can easily be solved, for example by the method of characteristics (Hildebrand 1962). For the normalised solution of (4.8) one then obtains

$$\begin{aligned} \psi &\equiv |A_1(t), A_2(t); \gamma_j; 00\rangle \\ &= \left(\frac{\alpha\beta}{\pi^2}\right)^{1/4} \exp\Big[-\frac{\alpha}{2}(x_1 - \langle x_1\rangle)^2 - \frac{\beta}{2}(x_2 - \langle x_2\rangle)^2 \\ &\quad + \frac{\mathrm{i}}{\hbar}\left(\langle p_1\rangle - \frac{m\alpha_1\gamma}{\hbar\gamma_2}\langle x_2\rangle\right)x_1 + \frac{\mathrm{i}}{\hbar}\left(\langle p_2\rangle - \frac{m\alpha_2\gamma}{\hbar\gamma_2}\langle x_1\rangle\right)x_2 + \frac{\mathrm{i}m\alpha_1\gamma}{\hbar^2\gamma_2}x_1x_2\Big] \end{aligned} \tag{4.10}$$

where $\langle x_1\rangle$ and $\langle x_2\rangle$ are the position coordinates for the classical orbits given in equations (4.2) and $\langle p_1\rangle$ and $\langle p_2\rangle$ are the corresponding canonical momenta, i.e. the expectation values of p_1 and p_2 in states (3.8). The parameters in equation (4.10) are defined by

$$\alpha \equiv \frac{m\alpha_1\gamma_4^2}{\hbar(m^2\alpha_1\alpha_2\gamma_2^2 + \gamma_3^2\gamma_4^2)}, \qquad \beta \equiv \frac{m\alpha_2\gamma_2^2}{\hbar(m^2\alpha_1\alpha_2\gamma_2^2 + \gamma_3^2\gamma_4^2)},$$

$$\gamma \equiv \frac{\hbar\gamma_2}{m\alpha_1}\,\frac{m^2\alpha_1\alpha_2\gamma_2(1-\gamma_1\gamma_2) - \gamma_1\gamma_3\gamma_4}{m^2\alpha_1\alpha_2\gamma_2^2 - \gamma_3^2\gamma_4^2}.$$

The corresponding probability density is

$$|\psi|^2 = (\alpha\beta/\pi^2)^{1/4}\exp[-\alpha(x_1 - \langle x_1\rangle)^2 - \beta(x_2 - \langle x_2\rangle)^2]. \tag{4.11}$$

The equi-probability contours are, therefore, given by the ellipses

$$Z_1^2/a^2 + Z_2^2/b^2 = 1 \tag{4.12}$$

where $Z_1 \equiv x_1 - \langle x_1 \rangle$ and $Z_2 \equiv x_2 - \langle x_2 \rangle$, $a^2 \equiv c/\alpha$, and $b^2 \equiv c/\beta$ and c is some real number.

From equations (4.12) and (4.2) it follows that when $\langle x_1 \rangle = \langle x_2 \rangle = 0$, i.e. for $|A_1| = |A_2| = 0$, the equi-probability contours are ellipses with their centres at the origin. Thus, in general, (4.12) is the equation for elliptical equi-probability contours with their centres moving on the classical orbits given in equation (4.2). Recalling the analysis of the motion of a charged particle in a constant uniform magnetic field (Landau and Lifshitz 1965, Johnson and Lippman 1949, Dulock and McIntosh 1966), it is clear that the coherent states (4.10) and (3.8) can be specialised to those for this system. In fact, these latter coherent states have already been obtained (Malkin and Man'Ko 1969, Feldman and Kahn 1970, Dodonov *et al* 1972).

5. Concluding remarks

An immediate application of the coherent states in equations (3.4) and (3.8) is in establishing the classical correspondence for the nuclear cranking model (Inglis 1954, Thouless and Valatin 1962, Gulshani and Rowe 1978a, b). This model has played an important role in the description of the rotational properties of certain nuclei at low as well as at high angular momenta (Bohr *et al* 1976, Johnson and Szymanski 1973). It is surmised that the coherent states (3.4) and (3.8) may be particularly useful in the description of nuclear high-spin states where the rotational motion may be treated quasiclassically. The coherent states (3.4) and (3.8) may also be of interest to the rotational motion of cluster-type systems, such as molecules and nuclear molecular systems reached in heavy-ion collisions (Greiner 1972).

For such an application it is necessary, however, to construct a multi-fermion angular momentum coherent state from that for a single particle given in equations (3.4) and (3.8). For a system of N identical non-interacting fermions moving in a common harmonic oscillator potential, such a construction is achieved in an obvious way as follows: introducing the label $\nu(1, 2, \ldots, N)$ for the particles, one replaces x_k, p_k, a_k and $a_k^\dagger$ in the previous sections by $x_{\nu k}$, $p_{\nu k}$, $a_{\nu k}$ and $a_{\nu k}^\dagger$ respectively, and sum over the particle index ν wherever these operators appear in the expresssions (3.4) and (3.8). Of course, one also replaces the single-particle oscillator state $|n_1, n_2\rangle$ in these expressions by an antisymmetrised product wavefunction (the Slater determinant) obtained from the occupied single-particle states $|n_{1\nu} n_{2\nu}\rangle$. That is, one simply makes the substitution

$$|n_1, n_2\rangle \to \frac{1}{\sqrt{N!}} \sum_P (-1)^p P |n_{11} n_{21}\rangle |n_{12} n_{22}\rangle \ldots |n_{1\nu} n_{2\nu}\rangle \ldots |n_{1N} n_{2N}\rangle$$

where $|n_{1\nu} n_{2\nu}\rangle$ is the νth occupied single-particle two-dimensional oscillator state and $\Sigma_P\ (-1)^p P$ is the well known antisymmetrising operator. It is then a straightforward matter to derive the results of the previous sections, using instead this multi-particle generalisation of the coherent states (3.4) and (3.8).

It is also of interest to construct the most general $\mathrm{N}(2) \otimes \mathrm{Sp}(4, \boldsymbol{R})$ coherent states that will describe rotational and vibrational motions as well as their intercoupling. Equally interesting is the generalisation of the two-dimensional angular momentum coherent states (3.4) and (3.8) to the three-dimensional case. Obviously the appropriate group in this case is the semi-direct product group N(3) ⊗ Sp(6, *R*), the dynamical group of the three-dimensional oscillator. In a following paper we shall address ourselves to these questions.

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